



## Generalization of Determining Area and Arc Length of Cosine Function in Polar Coordinate

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**ABSTRACT:** Polar coordinate is formed by the radius and the rotation angle. The radius indicates the distance of a point from the pole, while the rotation angle is the angle from which starting point moves to ending point relative to the axis-x. Because polar coordinate is affected by two variables, polar coordinate is defined on a plane. This research aims to find a general formula for determining area and length of a curve. The curve equation is a cosine function, which is circular and cardioid. The graph of circle is affected by constant and positive or negative sign on the cosine function. The constant determines the size of the graph, while positive or negative sign indicates position of the graph in the quadrant. Similarly, the graph of cardioid is affected by constant and positive or negative sign on the cosine function. The constant determines the size of the graph, while positive or negative sign indicates the direction of the cardioid. The area of circle is directly proportional to the square of constant. The arc length of circle is directly proportional to the constant. The area of cardioid is directly proportional to the square of constant. The arc length of cardioid is equal to eight times the constant. The general formula for the area and the arc length of the curve is used to predict their values. Simulations using several constant values are performed to practice the general formula for area and arc length of the circle and the cardioid.

**KEY WORDS:** Polar coordinate, Area, Arc length, Circle, Cardioid

### 1. INTRODUCTION

Polar coordinate is a pair of radius  $r$  and rotation angle  $\theta$ . The rotation angle  $\theta$  rotates completely by  $2\pi$ . The angle  $\theta$  is independent variable and  $r$  is dependent variable, so the curve equation is written as  $r = f(\theta)$  (Dawkins, 2007). Because there are two variables, polar coordinates are presented in a plane. The angle  $\theta$  will be processed by trigonometric functions, resulting in a regular graph. The resulting figure can be a circle or a cardioid (Vanberg et al., 2007). Polar coordinate be useful for solving definite integral problems that cannot or be difficult to solve using Cartesian coordinate. By using polar coordinate, definite integral problem can be solved easily and quickly. In addition, the discovery of polar coordinate can produce notation or constant that is very important in mathematics, for example  $\pi$  notation (Adam et al., 2022). Polar coordinate in statistics is useful for proving the property of the probability density function of normal distribution. The development of polar coordinate is in the form of cylindrical coordinate and spherical coordinate. These coordinates are useful for determining the volume of cylinder or sphere (Thomas et al., 1998).

Several researchs on polar coordinate have been conducted, including Barnet (2021), which discusses the arc length of a lemniscate. A lemniscate is a curve whose equation is the product of a constant and the square root of cosine or sine function. The lemniscate's shape resembles a rose. In Sudarno et al. (2025) discusses calculating the area and arc length of sine and cosine functions about simple functions. In that article hasn't been found general formula for determining area and arc length in any negative and positive real number.

This research extends generalization of general formula for determining area and arc length of the circle and the cardioid. The general formula is useful for simplifying determination of area and arc length of the curves. Furthermore, the general formula is useful for predicting their values for any positive and negative real number. The general formula is applied by conducting simulations on several positive integers.

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2. LITERATURE REVIEW

Cartesian and polar coordinates are coordinates that each have their own characteristics. But they have an interrelated relationship. In Cartesian coordinate, there is a plane area, whereas in polar coordinate, there is also a plane area. In Cartesian coordinate, there is a curve length, likewise the arc length is also found in polar coordinate [3]. Therefore, in this article we will discuss the area and the arc length in polar coordinate, which of course they are based on Cartesian coordinate.

2.1. The Relationship between Cartesian and Polar Coordinates

There is a reciprocal relationship between Cartesian and polar coordinates. Cartesian coordinate  $(x, y)$  and polar coordinate  $(r, \theta)$  are two coordinate systems for determining the position of a point on a two-dimensional plane. In Cartesian coordinates, the abscissa  $x$  is the perpendicular distance from the point to the axis- $y$  and the ordinate  $y$  is the perpendicular distance from the point to the axis- $x$ . Meanwhile, in polar coordinates, the radius  $r$  is the distance from the pole to the point and the angle  $\theta$  is the rotation angle from the positive axis- $x$  to the point. Both coordinate systems are related through trigonometric functions [5]. The relationship between Cartesian and polar coordinates can be tabulated in Table 1.

Table 1. The relationship between Cartesian and polar coordinates

Cartesian Coordinate	Polar Coordinate
$x$	$r$
$y$	$\theta$
$(x, y)$	$(r, \theta)$
$P(x, y)$	$P(r, \theta)$
$x = r \cos \theta$	$r = \sqrt{x^2 + y^2}$
$y = r \sin \theta$	$\theta = \tan^{-1} \left( \frac{y}{x} \right)$

2.2. The Area of Region in Polar Coordinate

We need to know the basic formula for area to easily calculate the area of a region in polar coordinate. The basic formula for area is the sector area formula. A sector is a portion of a circle [6]. We need to know the sector area, as stated in Proposition 1 below. Proposition 1.

Let there is a sector, then the area of a sector is  $A = \frac{1}{2} r^2 \theta$ , where  $r$  is the radius of the circle, and  $\theta$  is the rotation angle of the sector.

Proof:

If we define A as the area of a sector, B as the area of a circle, C as the rotation angle of a sector, and D as the rotation angle of a circle. Next, we use this comparison, namely

$$\frac{A}{B} = \frac{C}{D} \Leftrightarrow A = \frac{C}{D} B = \frac{\theta}{2\pi} \pi r^2 = \frac{1}{2} r^2 \theta .$$

Therefore, the area of a sector is  $A = \frac{1}{2} r^2 \theta$ . (1)

In polar coordinate, the independent variable is  $\theta$ , while the dependent variable is  $r$ . Therefore, the relationship between variables be written as  $r = f(\theta)$ . The notation states that the radius  $r$  is function of the rotation angle  $\theta$  [1]. Based on Proposition 1, we can find a method for calculating the area of a sector or region based on the function  $r = f(\theta)$ . The method for calculating the area of a sector is presented in Theorem 1.

Theorem 1.

Let  $\theta$  is the rotation angle of the point, whereas  $r$  is the distance from the pole to the point. If  $r = f(\theta)$ , where  $\alpha \leq \theta \leq \beta$ , then the sector area  $A$  be obtained by

$$A = \frac{1}{2} \int_{\theta=\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{\theta=\alpha}^{\beta} [f(\theta)]^2 d\theta. \tag{2}$$

Proof:

Based on Equation 1 that  $A = \frac{1}{2} r^2 \theta$ , then for  $r = f(\theta)$  with the rotation angle  $\alpha \leq \theta \leq \beta$ , the sector area is

$$A = \frac{1}{2} \int_{\theta=\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{\theta=\alpha}^{\beta} [f(\theta)]^2 d\theta.$$

### 2.3. The Arc Length of Curve in Polar Coordinate

The basis for determining arc length in polar coordinate is the same as in Cartesian coordinate. The arc length of a curve in polar coordinates is written in Theorem 2.

Theorem 2.

Let  $r = f(\theta)$  be a function about  $\theta$ . It has derivative and continuous on interval  $\alpha \leq \theta \leq \beta$ . The arc length of the curve  $r = f(\theta)$ , with the rotation angle  $\alpha \leq \theta \leq \beta$  is

$$L = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\theta. \tag{3}$$

Proof:

Let  $r = f(\theta)$ , where  $\alpha \leq \theta \leq \beta$ , then  $x = r \cos \theta = f(\theta) \cos \theta$  and  $y = r \sin \theta = f(\theta) \sin \theta$ , so  $x' = f'(\theta) \cos \theta - f(\theta) \sin \theta$  and  $y' = f'(\theta) \sin \theta + f(\theta) \cos \theta$ .

The arc length of the curve  $r = f(\theta)$  is

$$\begin{aligned} L &= \int_{\theta=\alpha}^{\beta} \sqrt{(x')^2 + (y')^2} d\theta \\ &= \int_{\theta=\alpha}^{\beta} \sqrt{(f'(\theta) \cos \theta - f(\theta) \sin \theta)^2 + (f'(\theta) \sin \theta + f(\theta) \cos \theta)^2} d\theta \\ &= \int_{\theta=\alpha}^{\beta} \sqrt{(f(\theta))^2 (\cos^2 \theta + \sin^2 \theta) + (f'(\theta))^2 (\cos^2 \theta + \sin^2 \theta)} d\theta \\ &= \int_{\theta=\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\theta \end{aligned}$$

Since  $\cos^2 \theta + \sin^2 \theta = 1$ . Therefore, if  $\alpha \leq \theta \leq \beta$ , then  $L = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + (r')^2} d\theta$ .

## 3. RESULT AND DISCUSSION

The following discussion applies area and arc length to curves in the shape of a circle and a cardioid. The result is presented starting with simple problem and expanding its to obtain a general formula. The general formula is presented in the form of a theorem. The theorem is useful as a tool for predicting the area or arc length of the curve.

### 3.1. Area and Arc Length of a Circle

#### A Circle in Quadrants One and Four

If given a curve  $r = a \cos \theta$ , where  $a$  is a positive real number and  $0 \leq \theta \leq 2\pi$ . The resulting graph is a circle located in quadrants one and four. The circle has center at  $(\frac{1}{2}a, 0)$  and radius is half  $a$ .

Problem 1.

Suppose there is a curve  $r = \cos \theta$ , where the rotation angle is  $0 \leq \theta \leq 2\pi$ . Determine area and arc length of the curve.

Solution:

We choose  $r_{\max} = 1 \iff 1 = \cos \theta \iff \theta = 0$ . We choose  $r_{\min} = 0 \iff 0 = \cos \theta \iff \theta = \frac{\pi}{2}$ .

Therefore, the rotation angle is  $0 \leq \theta \leq \frac{\pi}{2}$ .

The graph  $r = \cos \theta$  be presented at Figure 1 below.

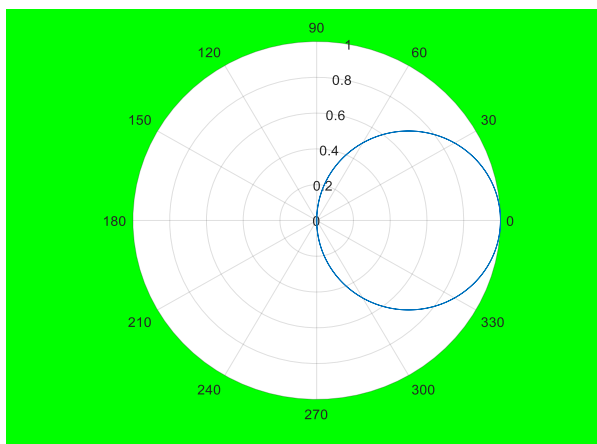


Figure 1. The graph  $r = \cos \theta$

The region area be created by the curve  $r = \cos \theta$  is

$$A = \int_{\theta=0}^{\pi/2} \cos^2 \theta \, d\theta = \frac{1}{2} \int_{\theta=0}^{\pi/2} [1 + \cos 2\theta] \, d\theta = \frac{1}{2} (\theta + \frac{1}{2} \sin 2\theta) \Big|_{\theta=0}^{\pi/2}$$

$$= \left[ \frac{1}{2} (\pi / 2) \right] = \frac{\pi}{4}$$

If  $r = \cos \theta$ , then  $r' = -\sin \theta$ , so that the arc length of the graph  $r$  is

$$S = 2 \int_{\theta=0}^{\pi/2} \sqrt{\cos^2 \theta + \sin^2 \theta} \, d\theta = 2\theta \Big|_{\theta=0}^{\pi/2} = \pi.$$

Problem 2.

Let there is the curve  $r = 2\cos \theta$ , where the rotation angle is  $0 \leq \theta \leq 2\pi$ . Determine area and arc length of the curve.

Solution:

We choose  $r_{\max} = 2 \iff 2 = 2\cos \theta \iff 1 = \cos \theta \iff \theta = 0.$

We choose  $r_{\min} = 0 \iff 0 = 2\cos \theta \iff 0 = \cos \theta \iff \theta = \frac{\pi}{2}.$

Therefore, the rotation angle is  $0 \leq \theta \leq \frac{\pi}{2}.$

The graph  $r = 2\cos \theta$  can be presented as Figure 2.

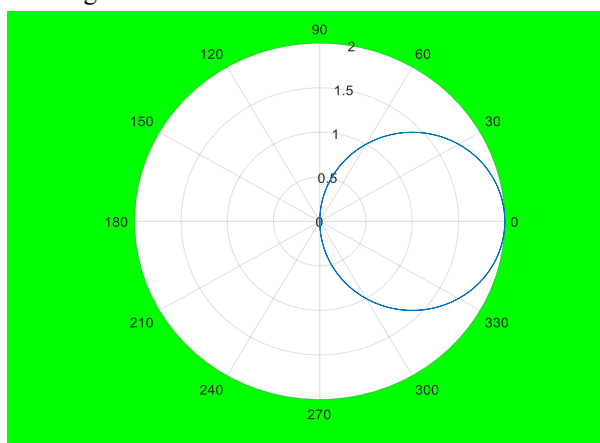


Figure 2. The graph  $r = 2\cos \theta$

The region area be made by the curve  $r = 2\cos \theta$  is

$$A = 4 \int_{\theta=0}^{\pi/2} \cos^2 \theta \, d\theta = 2 \int_{\theta=0}^{\pi/2} [1 + \cos 2\theta] \, d\theta = 2 (\theta + \frac{1}{2} \sin 2\theta) \Big|_{\theta=0}^{\pi/2}$$

$$= \left[ 2 (\pi / 2) \right] = \pi$$

If we take  $r = 2\cos \theta$ , then  $r' = -2\sin \theta$  and the arc length of the graph  $r$  is

$$S = 2 \int_{\theta=0}^{\pi/2} \sqrt{4 \cos^2 \theta + 4 \sin^2 \theta} d\theta = 4\theta \Big|_{\theta=0}^{\pi/2} = 2\pi.$$

Another expansion of the multiplication of positive integers by the cosine function is written in Theorem 3. This theorem is a generalization of area and arc length of the curve.

Theorem 3.

If there is a curve  $r = a \cos \theta$ , where  $a$  is positive real number,  $0 \leq \theta \leq \frac{\pi}{2}$ . Then

the area of the curve  $r$  is  $\frac{a^2}{4} \pi$ , and the arc length of the graph  $r$  is  $a\pi$ .

Proof:

We choose  $r_{\max} = a \iff a = a \cos \theta \iff 1 = \cos \theta \iff \theta = 0$ .

We choose  $r_{\min} = 0 \iff 0 = a \cos \theta \iff 0 = \cos \theta \iff \theta = \frac{\pi}{2}$ .

Therefore, the rotation angle is  $0 \leq \theta \leq \frac{\pi}{2}$ .

The area of the curve  $r = a \cos \theta$  is

$$\begin{aligned} A &= a^2 \int_{\theta=0}^{\pi/2} \cos^2 \theta d\theta = \frac{a^2}{2} \int_{\theta=0}^{\pi/2} [1 + \cos 2\theta] d\theta = \frac{a^2}{2} \left( \theta + \frac{1}{2} \sin 2\theta \Big|_{\theta=0}^{\pi/2} \right) \\ &= \left[ \frac{a^2}{2} \left( \frac{\pi}{2} \right) \right] = \frac{a^2}{4} \pi \end{aligned}$$

If the graph  $r = a \cos \theta$ , then  $r' = -a \sin \theta$ . Therefore, the arc length of the graph  $r = a \cos \theta$  is

$$S = 2 \int_{\theta=0}^{\pi/2} \sqrt{a^2 \cos^2 \theta + a^2 \sin^2 \theta} d\theta = 2a\theta \Big|_{\theta=0}^{\pi/2} = a\pi.$$

If we take  $a = 1, 2, \dots, 10$ , based on Theorem 3, we can determine area and arc length formed by the curve  $r = a \cos \theta$  as the Table 2.

Table 2. Simulation value of curve  $r = a \cos \theta$

$a$	$r = a \cos \theta$	$A$	$L$
1	$r = \cos \theta$	$\frac{\pi}{4}$	$\pi$
2	$r = 2 \cos \theta$	$\pi$	$2\pi$
3	$r = 3 \cos \theta$	$\frac{9}{4} \pi$	$3\pi$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$r = 9 \cos \theta$	$20 \frac{1}{4} \pi$	$9\pi$
10	$r = 10 \cos \theta$	$25\pi$	$10\pi$

### A Circle in Quadrants Two and Three

If given a curve  $r = -a \cos \theta$ , where  $-a$  is a negative real number and  $0 \leq \theta \leq 2\pi$ . The resulting graph is a circle located in quadrants two and three. The circle has center at  $(-\frac{1}{2}a, 0)$  and radius is half  $a$ .

Problem 3.

If there is the curve  $r = -\cos \theta$ , where the rotation angle is  $0 \leq \theta \leq 2\pi$ . Find area and arc length produced by the curve.

Solution:

We choose  $r_{\max} = 1 \iff 1 = -\cos \theta \iff \theta = \pi$ . We choose  $r_{\min} = 0 \iff 0 = -\cos \theta \iff \theta = \frac{\pi}{2}$ . Therefore, the rotation angle is

$$\frac{\pi}{2} \leq \theta \leq \pi.$$

The graph of circle  $r = -\cos\theta$  be shown by Figure 3 as follow.

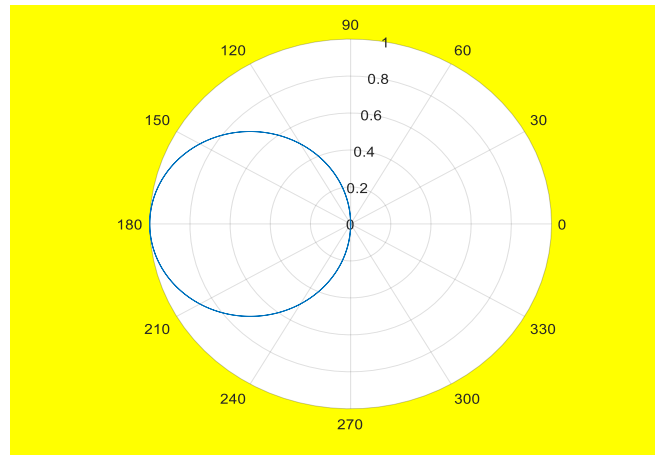


Figure 3. The graph  $r = -\cos\theta$

The area of the curve  $r = -\cos\theta$  is

$$\begin{aligned} A &= \int_{\theta=\pi/2}^{\pi} \cos^2 \theta \, d\theta = \frac{1}{2} \int_{\theta=\pi/2}^{\pi} [1 + \cos 2\theta] \, d\theta = \frac{1}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=\pi/2}^{\pi} \\ &= \left[ \frac{1}{2} \left( \frac{\pi}{2} \right) \right] = \frac{\pi}{4} \end{aligned}$$

If  $r = -\cos\theta$ , then  $r' = \sin\theta$ . Therefore, the arc length of the graph is

$$S = 2 \int_{\theta=\pi/2}^{\pi} \sqrt{\cos^2 \theta + \sin^2 \theta} \, d\theta = 2\theta \Big|_{\theta=\pi/2}^{\pi} = \pi.$$

Problem 4.

If there is the curve  $r = -2\cos\theta$ , where the rotation angle is  $0 \leq \theta \leq 2\pi$ . Find area and arc length be made by that curve.

Solution:

$$\text{We choose } r_{\max} = 2 \iff 2 = -2\cos\theta \iff -1 = \cos\theta \iff \theta = \pi.$$

$$\text{We choose } r_{\min} = 0 \iff 0 = -2\cos\theta \iff 0 = \cos\theta \iff \theta = \frac{\pi}{2}.$$

Therefore, the rotation angle is  $\frac{\pi}{2} \leq \theta \leq \pi$ .

The graph  $r = -2\cos\theta$  is presented at Figure 4.

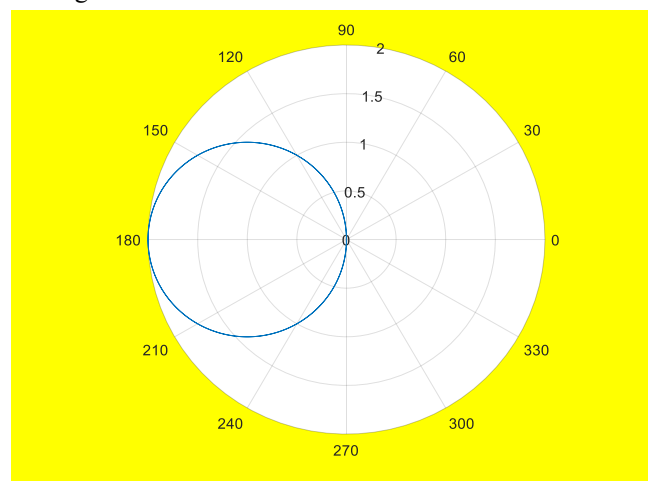


Figure 4. The graph  $r = -2\cos\theta$

The area of region created by the curve  $r = -2\cos\theta$  is

$$A = 4 \int_{\theta=\pi/2}^{\pi} \cos^2 \theta \, d\theta = 2 \int_{\theta=\pi/2}^{\pi} [1 + \cos 2\theta] \, d\theta = 2 \left( \theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=\pi/2}^{\pi} = [2(\pi/2)] = \pi$$

If we put on  $r = -2\cos\theta$ , then  $r' = 2\sin\theta$ . Therefore, the arc length of the graph is

$$S = 2 \int_{\theta=\pi/2}^{\pi} \sqrt{4\cos^2 \theta + 4\sin^2 \theta} \, d\theta = 4\theta \Big|_{\theta=\pi/2}^{\pi} = 2\pi .$$

Another expansion of the multiplication of negative integers by the cosine function is written in Theorem 4. This theorem is a generalization of area and arc length of the curve.

Theorem 4.

If there is a curve  $r = -a\cos\theta$ , where  $-a$  is negative real number,  $\frac{\pi}{2} \leq \theta \leq \pi$ . Then

the area of the curve is  $\frac{a^2}{4}\pi$ , and the arc length of the graph is  $a\pi$ .

Proof:

$$\text{We choose } r_{\max} = a \iff a = -a\cos\theta \iff -1 = \cos\theta \iff \theta = \pi .$$

$$\text{We choose } r_{\min} = 0 \iff 0 = -a\cos\theta \iff 0 = \cos\theta \iff \theta = \frac{\pi}{2} .$$

Therefore, the rotation angle is  $\frac{\pi}{2} \leq \theta \leq \pi$ .

The area of the curve  $r = -a\cos\theta$  is

$$A = a^2 \int_{\theta=\pi/2}^{\pi} \cos^2 \theta \, d\theta = \frac{a^2}{2} \int_{\theta=\pi/2}^{\pi} [1 + \cos 2\theta] \, d\theta = \frac{a^2}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=\pi/2}^{\pi} = \left[ \frac{a^2}{2} \left( \frac{\pi}{2} \right) \right] = \frac{a^2}{4}\pi$$

If the graph  $r = -a\cos\theta$ , then  $r' = a\sin\theta$ . Therefore, the arc length of the graph  $r = -a\cos\theta$  is

$$S = 2 \int_{\theta=\pi/2}^{\pi} \sqrt{a^2 \cos^2 \theta + a^2 \sin^2 \theta} \, d\theta = 2a\theta \Big|_{\theta=\pi/2}^{\pi} = a\pi .$$

If we take  $a = 1, 2, \dots, 10$ , based on Theorem 4, we could find both area and arc length made by the curve  $r = -a\cos\theta$  as shown in Table 3 below.

**Table 3. Simulation value of curve  $r = -a\cos\theta$**

$a$	$r = -a\cos\theta$	$A$	$L$
1	$r = -\cos\theta$	$\frac{\pi}{4}$	$\pi$
2	$r = -2\cos\theta$	$\pi$	$2\pi$
3	$r = -3\cos\theta$	$\frac{9}{4}\pi$	$3\pi$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$r = -9\cos\theta$	$20\frac{1}{4}\pi$	$9\pi$
10	$r = -10\cos\theta$	$25\pi$	$10\pi$

### 3.2. Area and Arc Length of a Cardioid A Cardioid with Direction to the Right

If given a curve  $r = \pm a + a \cos \theta$ , where  $a$  is a positive real number and  $0 \leq \theta \leq 2\pi$ . The resulting graph is a cardioid with direction to the right.

Problem 5.

Let there is a curve  $r = \pm 1 + \cos \theta$ , where the rotation angle is  $0 \leq \theta \leq 2\pi$ . Find area and arc length be shown by a curve  $r = \pm 1 + \cos \theta$ .

Solution:

We choose  $r_{\max} = 2 \iff 2 = 1 + \cos \theta \iff 1 = \cos \theta \iff \theta = 0$ .

We choose  $r_{\min} = 0 \iff 0 = 1 + \cos \theta \iff -1 = \cos \theta \iff \theta = \pi$ .

Therefore, we can choose the rotation angle is  $0 \leq \theta \leq \pi$ .

The graph of the cardioid is drawn below.

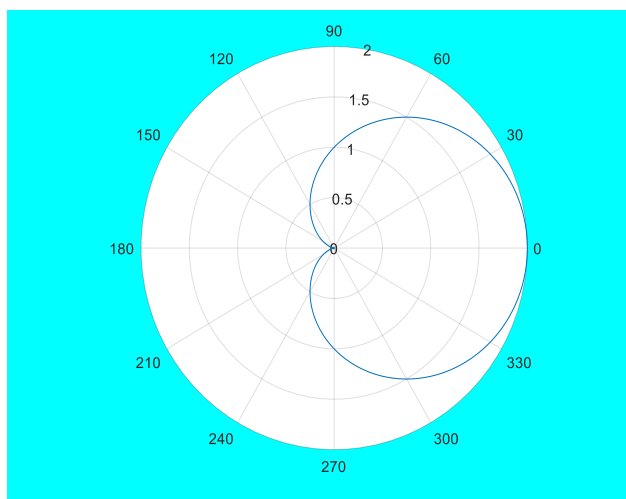


Figure 5. The graph  $r = \pm 1 + \cos \theta$

The area be generated by curve  $r = 1 + \cos \theta$  is

$$A = \int_{\theta=0}^{\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [1 + 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta$$

$$= (\theta + 2 \sin \theta + \frac{1}{2}(\theta + \frac{1}{2} \sin 2\theta)) \Big|_{\theta=0}^{\pi} = \left(\frac{3}{2}\pi\right)$$

The area be generated by curve  $r = -1 + \cos \theta$  is

$$A = \int_{\theta=0}^{\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [1 - 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta$$

$$= (\theta - 2 \sin \theta + \frac{1}{2}(\theta + \frac{1}{2} \sin 2\theta)) \Big|_{\theta=0}^{\pi} = \left(\frac{3}{2}\pi\right)$$

For the curve  $r = 1 + \cos \theta$ , then  $r' = -\sin \theta$ , so that the arc length be generated is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{4 \cos^2 \frac{1}{2} \theta} d\theta = 8 \sin \frac{1}{2} \theta \Big|_{\theta=0}^{\pi} = 8.$$

For the curve  $r = -1 + \cos \theta$ , then  $r' = -\sin \theta$ , so that the arc length be generated is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{4 \sin^2 \frac{1}{2} \theta} d\theta = -8 \cos \frac{1}{2} \theta \Big|_{\theta=0}^{\pi} = 8.$$

Problem 6.

Suppose there is a curve  $r = \pm 2 + 2 \cos \theta$ , where the rotation angle is  $0 \leq \theta \leq 2\pi$ . Determine area and arc length be made the curve.

Solution:

We choose  $r_{\max} = 4 \iff 4 = 2 + 2 \cos \theta \iff 1 = \cos \theta \iff \theta = 0$ .

We choose  $r_{\min} = 0 \iff 0 = 2 + 2 \cos \theta \iff -1 = \cos \theta \iff \theta = \pi$ .

Therefore, the rotation angle is  $0 \leq \theta \leq \pi$ .

The graph of the cardioid be presented as follow.

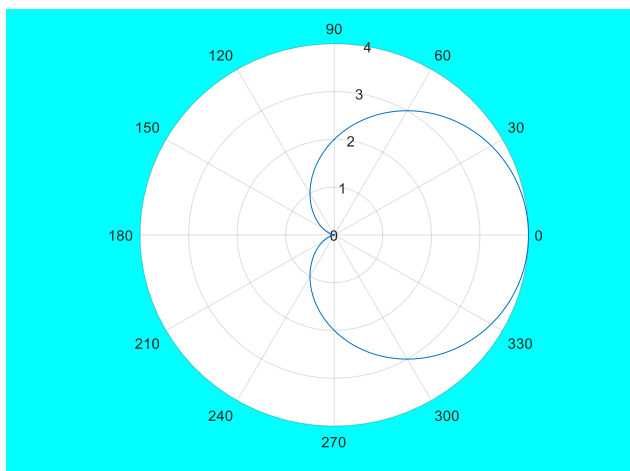


Figure 6. The graph  $r = \pm 2 + 2\cos \theta$

The area be generated by curve  $r = 2 + 2\cos \theta$  is

$$A = \int_{\theta=0}^{\pi} (4 + 8\cos \theta + 4\cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [4 + 8\cos \theta + 2(1 + \cos 2\theta)] d\theta$$

$$= (6\theta + 8\sin \theta + \sin 2\theta) \Big|_{\theta=0}^{\pi} = 6\pi$$

The area be generated by curve  $r = -2 + 2\cos \theta$  is

$$A = \int_{\theta=0}^{\pi} (4 - 8\cos \theta + 4\cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [4 - 8\cos \theta + 2(1 + \cos 2\theta)] d\theta$$

$$= (6\theta - 8\sin \theta + \sin 2\theta) \Big|_{\theta=0}^{\pi} = 6\pi$$

For the curve  $r = 2 + 2\cos \theta$ , then  $r' = -2\sin \theta$ , so the resulting arc length is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{4 + 8\cos \theta + 4\cos^2 \theta + 4\sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{16\cos^2 \frac{1}{2}\theta} d\theta$$

$$= 16 \sin \frac{1}{2}\theta \Big|_{\theta=0}^{\pi} = 16$$

For the curve  $r = -2 + 2\cos \theta$ , then  $r' = -2\sin \theta$ , so that the resulting arc length be

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{4 - 8\cos \theta + 4\cos^2 \theta + 4\sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{16\sin^2 \frac{1}{2}\theta} d\theta$$

$$= -16 \cos \frac{1}{2}\theta \Big|_{\theta=0}^{\pi} = 16$$

Another expansion of the multiplication of positive integers with cosine function to find area and arc length is written in Theorem 5. The theorem is generalization in determining area and length of a curve  $r = \pm a + a\cos \theta$ , for any positive real number  $a$ .

Theorem 5.

If there is a curve  $r = \pm a + a\cos \theta$ , where  $a$  is any positive real number,  $0 \leq \theta \leq 2\pi$ . Then

the area of the curve is  $\frac{3}{2}a^2\pi$ , and the arch length of the curve is  $8a$ .

Proof:

$$\text{We choose } r_{\max} = 2a \iff 2a = a + a\cos \theta \iff 1 = \cos \theta \iff \theta = 0.$$

$$\text{We choose } r_{\min} = 0 \iff 0 = a + a\cos \theta \iff -1 = \cos \theta \iff \theta = \pi.$$

Therefore, the selected rotation angle is  $0 \leq \theta \leq \pi$ .

The area produced by the curve  $r = a + a\cos \theta$  is

$$A = \int_{\theta=0}^{\pi} (a^2 + 2a^2 \cos \theta + a^2 \cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [a^2 + 2a^2 \cos \theta + \frac{1}{2}a^2(1 + \cos 2\theta)] d\theta$$

$$= (\frac{3}{2}a^2\theta + 2a^2 \sin \theta + \frac{1}{4}a^2 \sin 2\theta) \Big|_{\theta=0}^{\pi} = \frac{3}{2}a^2\pi$$

The area produced by the curve  $r = -a + a \cos \theta$  is

$$A = \int_{\theta=0}^{\pi} (a^2 - 2a^2 \cos \theta + a^2 \cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [a^2 - 2a^2 \cos \theta + \frac{1}{2}a^2(1 + \cos 2\theta)] d\theta$$

$$= (\frac{3}{2}a^2\theta - 2a^2 \sin \theta + \frac{1}{4}a^2 \sin 2\theta) \Big|_{\theta=0}^{\pi} = \frac{3}{2}a^2\pi$$

For the curve  $r = a + a \cos \theta$ , then  $r' = -a \sin \theta$ , so the resulting arc length is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{a^2 + 2a^2 \cos \theta + a^2 \cos^2 \theta + a^2 \sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{4 \cos^2 \frac{1}{2} \theta} d\theta$$

$$= 8a \sin \frac{1}{2} \theta \Big|_{\theta=0}^{\pi} = 8a$$

For the curve  $r = -a + a \cos \theta$ , then  $r' = -a \sin \theta$ , so the resulting arc length is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{a^2 - 2a^2 \cos \theta + a^2 \cos^2 \theta + a^2 \sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{4a^2 \sin^2 \frac{1}{2} \theta} d\theta$$

$$= -8a \cos \frac{1}{2} \theta \Big|_{\theta=0}^{\pi} = 8a$$

If we choose  $a = 1, 2, \dots, 10$ , based on Theorem 5, we can determine area and arc length formed by the curve  $r = \pm a + a \cos \theta$  as in Table 4 below.

**Table 4. Simulation value of curve  $r = \pm a + a \cos \theta$**

$a$	$r = \pm a + a \cos \theta$	$A$	$L$
1	$r = \pm 1 + \cos \theta$	$1\frac{1}{2}\pi$	8
2	$r = \pm 2 + 2 \cos \theta$	$6\pi$	16
3	$r = \pm 3 + 3 \cos \theta$	$13\frac{1}{2}\pi$	24
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$r = \pm 9 + 9 \cos \theta$	$121\frac{1}{2}\pi$	72
10	$r = \pm 10 + 10 \cos \theta$	$150\pi$	80

**A Cardioid with Direction to the Left**

A cardioid with direction to the left, it has equation  $r = \pm a - a \cos \theta$ , where  $-a$  is any negative real number and  $0 \leq \theta \leq 2\pi$  is full rotation angle.

Problem 7.

Suppose there is a curve  $r = \pm 1 - \cos \theta$ , where the rotation angle is  $0 \leq \theta \leq 2\pi$ . Find area and arc length be made by a curve  $r = \pm 1 - \cos \theta$ .

Solution:

We choose  $r_{\min} = 0 \iff 0 = 1 - \cos \theta \iff 1 = \cos \theta \iff \theta = 0$ .

We choose  $r_{\max} = 2 \iff 2 = 1 - \cos \theta \iff -1 = \cos \theta \iff \theta = \pi$ .

Therefore, the rotation angle is  $0 \leq \theta \leq \pi$ .

The graph of the cardioid is presented as Figure 7 below.

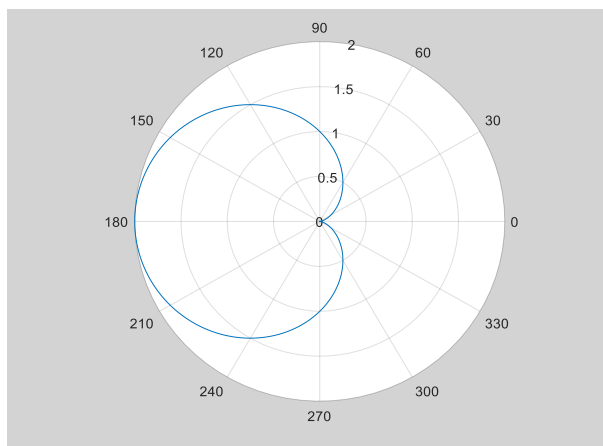


Figure 7. The graph  $r = \pm 1 - \cos \theta$

The area produced by curve  $r = 1 - \cos \theta$  is

$$A = \int_{\theta=0}^{\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [1 - 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta$$

$$= (\theta - 2 \sin \theta + \frac{1}{2}(\theta + \frac{1}{2} \sin 2\theta)) \Big|_{\theta=0}^{\pi} = \left(\frac{3}{2}\pi\right)$$

The area produced by curve  $r = -1 - \cos \theta$  is

$$A = \int_{\theta=0}^{\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [1 + 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta$$

$$= (\theta + 2 \sin \theta + \frac{1}{2}(\theta + \frac{1}{2} \sin 2\theta)) \Big|_{\theta=0}^{\pi} = \left(\frac{3}{2}\pi\right)$$

For the curve  $r = 1 - \cos \theta$ , then  $r' = \sin \theta$ , so that the resulting arc length is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{4 \sin^2 \frac{1}{2} \theta} d\theta$$

$$= -8 \cos \frac{1}{2} \theta \Big|_{\theta=0}^{\pi} = 8$$

For the curve  $r = -1 - \cos \theta$ , then  $r' = \sin \theta$ , so that the resulting arc length is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{4 \cos^2 \frac{1}{2} \theta} d\theta$$

$$= 8 \sin \frac{1}{2} \theta \Big|_{\theta=0}^{\pi} = 8$$

Problem 8.

Let there is a cardioid  $r = \pm 2 - 2 \cos \theta$ , where the rotation angle is  $0 \leq \theta \leq 2\pi$ . Determine area and arc length be made by that cardioid.

Solution:

We choose  $r_{\max} = 4 \iff 4 = 2 - 2 \cos \theta \iff -1 = \cos \theta \iff \theta = \pi$ .

We choose  $r_{\min} = 0 \iff 0 = 2 - 2 \cos \theta \iff 1 = \cos \theta \iff \theta = 0$ .

Therefore, the rotation angle is  $0 \leq \theta \leq \pi$ .

The graph of the cardioid be preseted as follow.

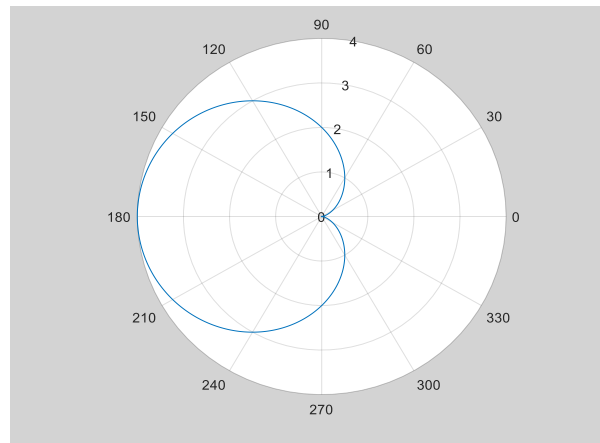


Figure 8. The cardioid  $r = \pm 2 - 2\cos\theta$

The area produced by cardioid  $r = 2 - 2\cos\theta$  is

$$\begin{aligned} A &= \int_{\theta=0}^{\pi} (4 - 8\cos\theta + 4\cos^2\theta) d\theta = \int_{\theta=0}^{\pi} [4 - 8\cos\theta + 2(1 + \cos 2\theta)] d\theta \\ &= (6\theta - 8\sin\theta + \sin 2\theta) \Big|_{\theta=0}^{\pi} = 6\pi \end{aligned}$$

The area produced by cardioid  $r = -2 - 2\cos\theta$  is

$$\begin{aligned} A &= \int_{\theta=0}^{\pi} (4 + 8\cos\theta + 4\cos^2\theta) d\theta = \int_{\theta=0}^{\pi} [4 + 8\cos\theta + 2(1 + \cos 2\theta)] d\theta \\ &= (6\theta + 8\sin\theta + \sin 2\theta) \Big|_{\theta=0}^{\pi} = 6\pi \end{aligned}$$

For curve  $r = 2 - 2\cos\theta$ , then  $r' = 2\sin\theta$ , so that the resulting arc length is

$$\begin{aligned} S &= 2 \int_{\theta=0}^{\pi} \sqrt{4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{16\sin^2\frac{1}{2}\theta} d\theta \\ &= -16\cos\frac{1}{2}\theta \Big|_{\theta=0}^{\pi} = 16 \end{aligned}$$

For curve  $r = -2 - 2\cos\theta$ , then  $r' = 2\sin\theta$ , so that the resulting arc length is

$$\begin{aligned} S &= 2 \int_{\theta=0}^{\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{16\cos^2\frac{1}{2}\theta} d\theta \\ &= 16\sin\frac{1}{2}\theta \Big|_{\theta=0}^{\pi} = 16 \end{aligned}$$

Another expansion of the multiplication of any negative real number with cosine function to find area and arc length is written in Theorem 6. The theorem is a generalization in determining area and length of a curve  $r = \pm a - a\cos\theta$ , for any negative real number  $-a$ .

Theorem 6.

If there is a curve  $r = \pm a - a\cos\theta$ , where  $-a$  is any negative real number,  $0 \leq \theta \leq 2\pi$ . Then

the area of the graph is  $\frac{3}{2}a^2\pi$ , and the arch length of the graph is  $8a$ .

Proof:

$$\text{We choose } r_{\max} = 2a \iff 2a = a - a\cos\theta \iff -1 = \cos\theta \iff \theta = \pi.$$

$$\text{We choose } r_{\min} = 0 \iff 0 = a - a\cos\theta \iff 1 = \cos\theta \iff \theta = 0.$$

Therefore, the rotation angle selected is  $0 \leq \theta \leq \pi$ .

The area produced by curve  $r = a - a\cos\theta$  is

$$\begin{aligned} A &= \int_{\theta=0}^{\pi} (a^2 - 2a^2\cos\theta + a^2\cos^2\theta) d\theta = \int_{\theta=0}^{\pi} [a^2 - 2a^2\cos\theta + \frac{1}{2}a^2(1 + \cos 2\theta)] d\theta \\ &= (\frac{3}{2}a^2\theta - 2a^2\sin\theta + \frac{1}{4}a^2\sin 2\theta) \Big|_{\theta=0}^{\pi} = \frac{3}{2}a^2\pi \end{aligned}$$

The area produced by curve  $r = -a - a\cos\theta$  is

$$A = \int_{\theta=0}^{\pi} (a^2 + 2a^2 \cos \theta + a^2 \cos^2 \theta) d\theta = \int_{\theta=0}^{\pi} [a^2 + 2a^2 \cos \theta + \frac{1}{2}a^2(1 + \cos 2\theta)] d\theta$$

$$= (\frac{3}{2}a^2\theta + 2a^2 \sin \theta + \frac{1}{4}a^2 \sin 2\theta) \Big|_{\theta=0}^{\pi} = \frac{3}{2}a^2\pi$$

For the curve  $r = a - a \cos \theta$ , then  $r' = a \sin \theta$ , so that the resulting arc length is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{a^2 - 2a^2 \cos \theta + a^2 \cos^2 \theta + a^2 \sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{4a^2 \sin^2 \frac{1}{2}\theta} d\theta$$

$$= -8a \cos \frac{1}{2}\theta \Big|_{\theta=0}^{\pi} = 8a$$

For the curve  $r = -a - a \cos \theta$ , then  $r' = a \sin \theta$ , so that the resulting arc length is

$$S = 2 \int_{\theta=0}^{\pi} \sqrt{a^2 + 2a^2 \cos \theta + a^2 \cos^2 \theta + a^2 \sin^2 \theta} d\theta = 2 \int_{\theta=0}^{\pi} \sqrt{4\cos^2 \frac{1}{2}\theta} d\theta$$

$$= 8a \sin \frac{1}{2}\theta \Big|_{\theta=0}^{\pi} = 8a$$

If we take  $a = 1, 2, \dots, 10$ , based on Theorem 6, we could find area and arc length formed by the curve  $r = \pm a - a \cos \theta$  as in Table 5 below.

**Table 5. Simulation value of curve  $r = -a - a \cos \theta$**

$a$	$r = \pm a + a \cos \theta$	$A$	$L$
1	$r = \pm 1 - \cos \theta$	$1\frac{1}{2}\pi$	8
2	$r = \pm 2 - 2 \cos \theta$	$6\pi$	16
3	$r = \pm 3 - 3 \cos \theta$	$13\frac{1}{2}\pi$	24
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	$r = \pm 9 - 9 \cos \theta$	$121\frac{1}{2}\pi$	72
10	$r = \pm 10 - 10 \cos \theta$	$150\pi$	80

#### 4. CONCLUSION

A point in polar coordinate is a pair of radius  $r$  and rotation angle  $\theta$ , with the notation  $(r, \theta)$ . A set of points will form the curve  $r = f(\theta)$ . A graph  $r = \pm a \cos \theta$ , where  $a$  is any positive real number,  $0 \leq \theta \leq 2\pi$  shape the circle. The circle  $r = \pm a \cos \theta$  has both the area of circle is  $\frac{a^2}{4}\pi$ , and the arc length of circle is  $a\pi$ . A graph  $r = \pm a \pm a \cos \theta$ , where  $a$  is any positive real number,  $0 \leq \theta \leq 2\pi$  shape the cardioid. The cardioid  $r = \pm a \pm a \cos \theta$  has both the area of the cardioid is  $\frac{3}{2}a^2\pi$ , and the arch length of the cardioid is  $8a$ . The general formula for area and arc length of the curve can be used as a tool to predict their values.

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